Grade 11/12 Math Circles
October 1, 2023
Digital Signal Processing - Solutions

Exercise 1

Evaluate the following sums:

a) $\sum_{k=1}^{4} k^2$

b) $\sum_{k=1}^{5} (2k+1)$

c) $\sum_{n=3}^{5} n(n+1)$

d) $\sum_{j=-1}^{1} (j+1)$

Exercise 1 Solution

a) $\sum_{k=1}^{4} k^2 = 1^2 + 2^2 + 3^2 + 4^2 = 1 + 4 + 9 + 16 = 30$

b) $\sum_{k=1}^{5} (2k+1) = (2+1) + (4+1) + (6+1) + (8+1) + (10+1) = 3 + 5 + 7 + 9 + 11 = 35$

c) $\sum_{n=3}^{5} n(n+1) = 3(3+1) + 4(4+1) + 5(5+1) = 12 + 20 + 30 = 62$

d) $\sum_{j=-1}^{1} (j+1) = (-1+1) + (0+1) + (1+1) = 0 + 1 + 2 = 3$

Exercise 2

Write each of the following finite signals as a weighted sum of shifted delta functions.

a)

n	0	1	2	3	4
$x[n]$	1	3	5	-1	2

b) $x[n] = \begin{cases} 0 & \text{if } n < 0 \\ n & \text{if } 0 \leq n \leq 3 \\ 0 & \text{if } n > 3 \end{cases}$

For an extra challenge: see if you can write this using sigma ($\sum$) notation!

Exercise 2 Solution

a) $x[n] = \delta[n] + 3\delta[n-1] + 5\delta[n-2] - \delta[n-3] + 2\delta[n-4]$

b) $x[n] = 0 \cdot \delta[n] + 1\delta[n-1] + 2\delta[n-2] + 3\delta[n-3] = \sum_{k=1}^{3} k \cdot \delta[n-k]$

Exercise 3

Determine the impulse response of the following digital filters.

a) $y[n] = x[n] - x[n-1]$

b) $y[n] = \text{median}\,(x[n], x[n-1], x[n-2])$

Exercise 3 Solution

a) $h[n] = \delta[n] - \delta[n-1] = \begin{cases} 1 & \text{if } n = 0 \\ -1 & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$

b) $h[n] = 0$

Exercise 4

Determine which of the following digital filters are causal.

a) $y[n] = \sum_{k=1}^{N} k \cdot x[n-k]$

b) $y[n] = \text{median}\left(x[n], x[n-1], x[n-2]\right)$

c) $y[n] = \left(x[n]\right)^2$

d) $y[n] = \frac{x[n]+x[n+1]}{2}$

Exercise 4 Solution

a) causal

b) causal

c) causal

d) not causal

Exercise 5

Find a counterexample to show that the following filters are non-linear.

a) $y[n] = 3x[n] + 5$

b) **CHALLENGE:** $y[n] = \text{median}\left(x[n], x[n-1], x[n-2]\right)$

Exercise 5 Solution

a) Let $x_1[n] = 1$ and $x_2[n] = 2$. Then $y_1[n] = 3(1) + 5 = 8$ and $y_2[n] = 3(2) + 5 = 11$.

Now let $z[n] = x_1[n] + x_2[n]$. Then the filter output is

$$\begin{aligned} 3z[n] + 5 &= 3(1 + 2) + 5 \\ &= 9 + 5 \\ &= 14 \neq y_1[n] + y_2[n] = 8 + 11 = 19. \end{aligned}$$

The filter is non-linear.

b) Let $x_1[n]$ and $x_2[n]$ be defined as follows (and assumed to be equal to zero everywhere else:

n	0	1	2
$x_1[n]$	1	2	3
$x_2[n]$	1	-1	-1

We see that $y_1[2] = \text{median}(x_1[2], x_1[1], x_1[0]) = 2$,
and $y_2[2] = \text{median}(x_2[2], x_2[1], x_2[0]) = -1$.

Now if we define $z[n] = x_1[n] + x_2[n]$, and define $w[n] = \text{median}(z[n], z[n-1], z[n-2])$, then

$$\begin{aligned} w[2] &= \text{median}(z[2], z[1], z[0]) \\ &= \text{median}(x_1[2] + x_2[2], x_1[1] + x_2[1], x_1[0] + x_2[0]) \\ &= \text{median}(2, 1, 2) \\ &= 2 \neq y_1[2] + y_2[2] = 2 - 1 = 1. \end{aligned}$$

Therefore the median filter is non-linear.

Exercise 6

Consider an input signal $x[n]$ and a time delayed signal $x[n-n_0]$. Use this to determine whether or not the following digital filters are time-invariant.

a) $y[n] = \frac{1}{2}(x[n] + x[n-1])$

b) $y[n] = x[n^2]$

Exercise 6 Solution

a) Consider the time delayed signal $z[n] = x[n-n_0]$. With $z[n]$ as input, the filter output is

$$\begin{aligned}\frac{1}{2}(z[n] + z[n-1]) &= \frac{1}{2}(x[n-n_0] + x[n-1-n_0]) \\ &= \frac{1}{2}(x[n-n_0] + x[n-n_0-1]) \\ &= y[n-n_0].\end{aligned}$$

Therefore this is a time-invariant filter.

b) Once again, let $z[n] = x[n-n_0]$. The filter output is

$$\begin{aligned}z[n^2] &= x[n^2 - n_0] \\ &\neq y[n-n_0] = x[(n-n_0)^2].\end{aligned}$$

Therefore this is not a time-invariant filter.